

On the discrete spectrum of spin-orbit Hamiltonians with singular interactions

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Abstract

We give a variational proof of the existence of infinitely many bound states below the continuous spectrum for spin-orbit Hamiltonians (including the Rashba and Dresselhaus cases) perturbed by measure potentials thus extending the results of J. Brüning, V. Geyler, K. Pankrashkin: J. Phys. A: Math. Gen. **40** (2007) F113–F117.

1 Introduction

It is well known that perturbations of the Laplacian by sufficiently localized potentials produce only a finite number of negative eigenvalues, and only long-range potentials can produce an infinite discrete spectrum, see Section XII.3 in [13]. This does not hold any more if one considers perturbations of magnetic Schrödinger operators, where compactly supported perturbations demonstrate a non-classical behavior [11, 14].

Recently such questions have become of interest in the study of operators related to the spintronics. Namely, in [8] it was emphasized that perturbations of the Rashba and Dresselhaus Hamiltonians

$$H_R = \begin{pmatrix} p^2 & \alpha(p_y + ip_x) \\ \alpha(p_y - ip_x) & p^2 \end{pmatrix},$$
$$H_D = \begin{pmatrix} p^2 & -\alpha(p_x + ip_y) \\ -\alpha(p_x - ip_y) & p^2 \end{pmatrix},$$

(α is a constant expressing the strength of the spin-orbit coupling [6, 12, 16]) by localized spherically symmetric negative potentials produce infinitely many

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eigenvalues below the continuous spectrum; in the justification some approximations have been used. In the paper [4] we gave a rigorous proof of this effect for rather general negative potentials without any symmetry conditions. In the present note we are going to extend these results and to obtain similar estimates for operators of the form $H = H_0 + \nu$, where H_0 is an unperturbed spin-orbit Hamiltonian and ν is a measure (whose support can have the zero Lebesgue measure). In particular, for the Rashba and Dresselhaus Hamiltonians we show that negative perturbations supported by curves always produce infinitely many bound states below the threshold.

2 Definition of Hamiltonians

Denote by \mathcal{H} the Hilbert space $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ of two-dimensional spinors; by \mathcal{F} we denote the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$; then $\mathcal{F}_2 := \mathcal{F} \otimes 1$ is the Fourier transform in \mathcal{H} . Let H_0 be the self-adjoint operator in \mathcal{H} whose Fourier transform $\hat{H}_0 := \mathcal{F}_2 H_0 \mathcal{F}_2^{-1}$ is the multiplication by the matrix

$$\hat{H}_0(p) = \begin{pmatrix} p^2 & A(p) \\ \overline{A(p)} & p^2 \end{pmatrix}, \quad p \in \mathbb{R}^2, \quad (1)$$

where A is a continuous complex function on \mathbb{R}^2 . We assume

$$\limsup_{p \rightarrow \infty} \frac{|A(p)|}{p^2} < 1. \quad (2)$$

Clearly, H_0 has no discrete spectrum; its spectrum is the union of images of two functions λ_{\pm} (dispersion laws): $\lambda_{\pm}(p) = p^2 \pm |A(p)|$, hence $\text{spec } H_0 = [\kappa, +\infty)$, where $\kappa := \inf\{p^2 - |A(p)| : p \in \mathbb{R}^2\} > -\infty$. Moreover, there is a unitary matrix $M(p)$ depending continuously on $p \in \mathbb{R}^2$ such that

$$M(p) \hat{H}_0(p) M^*(p) = \begin{pmatrix} \lambda_+(p) & 0 \\ 0 & \lambda_-(p) \end{pmatrix}, \quad p \in \mathbb{R}^2. \quad (3)$$

Denote $S := \{p \in \mathbb{R}^2 : \lambda_-(p) = \kappa\}$; this is a non-empty compact set. We will assume that

$$\text{the function } |A(p)| \text{ is of class } C^2 \text{ in a neighborhood of } S. \quad (4)$$

For the Rashba and Dresselhaus Hamiltonians one has $\kappa = -\alpha^2/4$ and S is the circle $\{p : 2|p| = |\alpha|\}$, and the condition (4) is obviously satisfied.

The two conditions (2) and (4) guarantee that for any $p_0 \in S$ there is a constant $c(p_0) > 0$ such that

$$0 \leq \lambda_-(p) - \kappa \leq c(p_0)|p - p_0|^2 \quad \text{for all } p \in \mathbb{R}^2. \quad (5)$$

We fix a positive Radon measure m on \mathbb{R}^2 and a bounded Borel measurable function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that there exist constants $a \in (0, 1)$ and $b > 0$ with

$$\int_{\mathbb{R}^2} (1 + |h(x)|^2) |f(x)|^2 m(dx) \leq a \int_{\mathbb{R}^2} |\nabla f(x)|^2 dx + b \int_{\mathbb{R}^2} |f(x)|^2 dx \quad (6)$$

for all f from the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. Denote $\nu := hm$. The assumption (6) is satisfied for any (bounded) h if m belongs to the Kato class measures, i.e.

$$\lim_{\varepsilon \rightarrow 0+} \sup_{x \in \mathbb{R}^2} \int_{|x-y| < \varepsilon} \log \frac{1}{|x-y|} |m(dy)| = 0.$$

This holds, for example, for δ -type measures concentrated on C^1 curves under some regularity conditions (this conditions are satisfied for compact curves and straight lines), see Section 4 in [3] for details.

Our aim now is to give a rigorous definition of the operator given by the formal expression

$$H = H_0 + \nu. \quad (7)$$

As $\mathcal{S}(\mathbb{R}^2)$ is dense in the Sobolev space $H^1(\mathbb{R}^2)$ there exists a unique linear bounded transformation J defined by

$$J : H^1(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2, m), \quad Jf = f \text{ for all } f \in \mathcal{S}(\mathbb{R}^2).$$

We denote $J_2 := J \otimes 1$; this is an operator acting from $H^1(\mathbb{R}^2) \otimes \mathbb{C}^2$ to $L^2(\mathbb{R}^2, m) \otimes \mathbb{C}^2$. For a continuous function f we denote the corresponding equivalence classes in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}, m)$ by the same letter f .

Now note that the operator H_0 can be presented as

$$H_0 = -\Delta_2 + L, \quad -\Delta_2 := -\Delta \otimes 1, \quad \widehat{L} := \mathcal{F}_2 L \mathcal{F}_2^{-1} = \begin{pmatrix} 0 & A(p) \\ \frac{0}{A(p)} & 0 \end{pmatrix}.$$

(Here Δ is the scalar two-dimensional Laplacian.)

Lemma 1. *The operator L is relatively bounded with respect to Δ_2 , and $\|L(-\Delta_2 + \lambda)^{-1}\| < 1$ for $\lambda \rightarrow +\infty$.*

Proof. The relative boundedness is obvious, so we only prove the norm estimate. Passing to the Fourier transform, we need to show

$$\sup_{p \in \mathbb{R}^2} \left| \frac{A(p)}{p^2 + \lambda} \right| < 1, \quad \lambda \rightarrow +\infty. \quad (8)$$

By (2), there exist $a < 1$ and $R > 0$ such that $|A(p)|/p^2 \leq a$ for all p with $|p| > R$. Then obviously one has

$$\left| \frac{A(p)}{p^2 + \lambda} \right| < a, \quad |p| > R, \quad \lambda > 0. \quad (9)$$

Due to the continuity of A there exists $C > 0$ with $|A(p)| \leq C$ for $|p| \leq R$. Then obviously there exists $\lambda_0 > 0$ such that

$$\left| \frac{A(p)}{p^2 + \lambda} \right| \leq C\lambda^{-1} < a, \quad |p| \leq R, \quad \lambda > \lambda_0. \quad (10)$$

Combining (9) with (10) we arrive at (8). \square

Eq. (6) and Lemma 1 imply that, by the KLMN theorem, the quadratic form

$$\begin{aligned} q(f, g) &= q_0(f, g) + \nu(f, g), \\ \nu(f, g) &:= \langle hJ_2f, J_2g \rangle_{L^2(\mathbb{R}^2) \otimes \mathbb{C}^2} \equiv \int_{\mathbb{R}^2} \langle J_2f(x), J_2g(x) \rangle_{\mathbb{C}^2} \nu(dx), \\ \nu(dx) &= h(x)m(dx), \end{aligned}$$

where q_0 is the quadratic form associated with H_0 , is semibounded below and closed on $H^1(\mathbb{R}^2) \otimes \mathbb{C}^2$ and hence defines a certain self-adjoint operator H semibounded below. If the measure ν is absolutely continuous with respect to the Lebesgue measure, i.e. $\nu(dx) = V(x)dx$ with a certain locally integrable function V , then the above procedure gives the usual form sum $H = H_0 + V$, so one preserves the same notation for the general case, $H_0 + \nu := H$.

Repeating the procedure from Section 2 in [3] one can express the resolvent of H through the resolvent H_0 . Combining this with the explicit expressions for the Green function for H_0 [5] one can obtain rather detailed formulas for the Green function of H , but we will not need this below. The following assertion about the spectral properties of H is important for us.

Theorem 2. *If the measure m is finite, then the essential spectra of H_0 and H coincide.*

Proof. The paper [2] deals with a rather detailed spectral analysis of operators defined by the sums of quadratic forms. According to Theorem 7 in [2] we only need to prove that the operator $J_2(H_0 - z)^{-1} : L^2(\mathbb{R}^2) \otimes \mathbb{C}^2 \rightarrow L^2(\mathbb{R}^2, m) \otimes \mathbb{C}^2$ is compact. Note that for sufficiently large λ the operator $1 + L(-\Delta_2 + \lambda)^{-1}$ has a bounded inverse defined everywhere due to lemma 1. Hence $(H_0 + \lambda)^{-1} = (-\Delta_2 + \lambda)^{-1}(1 + L(-\Delta_2 + \lambda)^{-1})^{-1}$, and the compactness of $J_2(H_0 + \lambda)^{-1}$ would follow from the compactness of $J(-\Delta + \lambda)^{-1} : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2, m)$ as $J_2(-\Delta_2 + \lambda)^{-1} \equiv (J(-\Delta + \lambda)^{-1}) \otimes 1$. At the same time, $J(-\Delta + \lambda)^{-1} = B^*$,

$$B : L^2(\mathbb{R}^2, m) \rightarrow L^2(\mathbb{R}^2), \quad Bf(x) = \int_{\mathbb{R}^2} G_0(x, y; \lambda) f(y) m(dy) \quad \text{a.e.},$$

where G_0 is the Green function of the two-dimensional Laplacian. In Lemma 2.3 in [3] it was shown that B is compact. Therefore, $J(-\Delta + \lambda)^{-1}$ is also compact, and the theorem is proved. \square

From now on we assume that the measure m is finite.

3 General perturbations

Below for a distribution f we denote the Fourier transform of f by \widehat{f} . We call an Hermitian $n \times n$ matrix C *positive definite* if for any non-zero $\xi \in \mathbb{C}^n$ there holds $\langle \xi, C\xi \rangle > 0$, and *positive semi-definite* if the above equality is non-strict. By analogy one introduces *negative definite* and *negative semi-definite* matrices.

The following result differs only in minor details from the main result in [5]

Theorem 3. Let $N \in \mathbb{N}$; assume that the Fourier transform $\widehat{\nu}$ satisfies the following condition: there are N points $p_1, \dots, p_N \in S$ such that the matrix $(\widehat{\nu}(p_j - p_k))_{j,k=1}^N$ is negative definite. Then H has at least N eigenvalues, counting multiplicity, below κ .

Proof. According to the max-min principle, it is sufficient to show that we can find N vectors $\Psi_m \in \mathcal{H}$, $m = 1, \dots, N$, such that the matrix with the entries $(q - \kappa)(\Psi_j, \Psi_k)$, $j, k = 1, \dots, N$, $(q - \kappa)(\Phi, \Psi) := q(\Phi, \Psi) - \kappa\langle\Phi, \Psi\rangle$ is negative definite.

Set $f_a(x) := \exp\left(-\frac{1}{2}|x|^a\right)$, $x \in \mathbb{R}^2$, with $a > 0$. Clearly, $f_a \in H^1(\mathbb{R}^2)$. It is observed in [17] that

$$\int_{\mathbb{R}^2} |\nabla f_a(x)|^2 dx = \frac{\pi}{2}a. \quad (11)$$

Furthermore, by the Lebesgue dominated convergence theorem,

$$\lim_{a \rightarrow 0+} \int_{\mathbb{R}^2} |f_a(x)|^2 \nu(dx) = e^{-1} \int_{\mathbb{R}^2} \nu(dx).$$

Let \widehat{f}_a be the Fourier transform of f_a . Take spinors Ψ_j such that their Fourier transforms $\widehat{\Psi}_j$ are of the form $\widehat{\Psi}_j(p) = M(p)\psi_j(p)$, where

$$\psi_j(p) = \begin{pmatrix} 0 \\ \widehat{f}_a(p - p_j) \end{pmatrix} \quad (12)$$

and $M(p)$ is taken from (3). We show that if a is sufficiently small, then the matrix $(q - \kappa)(\Psi_j, \Psi_k)$ is negative definite. For this purpose it is sufficient to show that

$$\lim_{a \rightarrow 0} (q_0 - \kappa)(\Psi_j, \Psi_k) = 0, \quad (13)$$

$$\lim_{a \rightarrow 0} \nu(\Psi_j, \Psi_k) = 2\pi e^{-1} \widehat{\nu}(p_j - p_k) \quad (14)$$

for all j and k .

By definition of Ψ_j one has

$$\begin{aligned} |(q_0 - \kappa)(\Psi_j, \Psi_k)| &= \left| \int_{\mathbb{R}^2} (\lambda_-(p) - \kappa) \widehat{f}_a(p - p_j) \overline{\widehat{f}_a(p - p_k)} dp \right| \\ &\leq \sqrt{\int_{\mathbb{R}^2} (\lambda_-(p) - \kappa) |\widehat{f}_a(p - p_j)|^2 dp} \\ &\quad \times \sqrt{\int_{\mathbb{R}^2} (\lambda_-(p) - \kappa) |\widehat{f}_a(p - p_k)|^2 dp}. \end{aligned}$$

On the other hand, by (5) and (11) one has

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^2} (\lambda_-(p) - \kappa) |\widehat{f}_a(p - p_j)|^2 dp \\ &\leq c(p_j) \int_{\mathbb{R}^2} (p - p_j)^2 |\widehat{f}_a(p - p_j)|^2 dp = c(p_j) \int_{\mathbb{R}^2} p^2 |\widehat{f}_a(p)|^2 dp \\ &= c(p_j) \int_{\mathbb{R}^2} |\nabla f_a(x)|^2 dx = \frac{\pi}{2} c(p_j) a, \end{aligned}$$

which proves (13). As for (14), one has

$$\begin{aligned}\nu(\Psi_j, \Psi_k) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \widehat{\nu}(p-q) \langle \widehat{\Psi}_j(p), \widehat{\Psi}_k(q) \rangle_{\mathbb{C}^2} dp dq \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \widehat{\nu}(p-q) \langle \widehat{\psi}_j(p), \widehat{\psi}_k(q) \rangle_{\mathbb{C}^2} dp dq.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \widehat{\nu}(p-q) \langle \widehat{\psi}_j(p), \widehat{\psi}_k(q) \rangle_{\mathbb{C}^2} dp dq \\ = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \widehat{\nu}(p-q) \overline{\widehat{f}_a(p-p_j)} \widehat{f}_a(q-p_k) dp dq \\ = \int_{\mathbb{R}^2} e^{i\langle p_j-p_k, x \rangle} |f_a(x)|^2 \nu(dx) \xrightarrow{a \rightarrow 0} 2\pi e^{-1} \widehat{\nu}(p_j-p_k),\end{aligned}$$

and the theorem is proved. \square

The above theorem allows to generalize some classical results. For example, the condition

$$\int_{\mathbb{R}^2} \nu(dx) \equiv 2\pi \widehat{\nu}(0) < 0.$$

guarantees that H has at least one eigenvalue below κ .

To formulate further corollaries, by $\#S$ we denote the number of points in S , if S is finite, and ∞ , otherwise.

Corollary 4. *Let $\nu \leq 0$ and $\text{supp } \nu$ have a positive Lebesgue measure. Then H has at least $\#S$ eigenvalues below κ .*

Proof. We show that the matrix $(\widehat{\nu}(p_j-p_k))_{j,k=1}^N$ is negative definite for any choice of pairwise different points $p_1, \dots, p_N \in \mathbb{R}^2$. By the Bochner theorem, $-\sum_{j,k} \widehat{\nu}(p_j-p_k) \bar{\xi}_j \xi_k \geq 0$ for any $(\xi_j) \in \mathbb{C}^N$ and it remains to note that $\sum_{j,k} \widehat{\nu}(p_j-p_k) \bar{\xi}_m \xi_n \neq 0$ for $(\xi_j) \neq 0$. In fact, if $\sum_{j,k} \widehat{\nu}(p_j-p_k) \bar{\xi}_j \xi_k = 0$, then

$$\int_{\mathbb{R}^2} \left| \sum_j \xi_j e^{i\langle p_j, x \rangle} \right|^2 \nu(dx) = 0;$$

therefore, $\sum_j \xi_j e^{i\langle p_j, x \rangle} = 0$ on the support of ν . As the exponents $e^{i\langle p_j, x \rangle}$ are real analytic, and $\sum_j \xi_j e^{i\langle p_j, x \rangle} = 0$ on a set of positive Lebesgue measure, the equality $\sum_j \xi_j e^{i\langle p_j, x \rangle} = 0$ is valid everywhere on \mathbb{R}^2 . On the other hand, $e^{i\langle p_j, x \rangle}$ are linearly independent, and we obtain $\xi_j = 0$ for all j . \square

Corollary 4 shows the existence of infinitely many eigenvalues below κ for perturbations of the Rashba and Dresselhaus Hamiltonians by negative measures with support of positive Lebesgue measure, i.e. given by negative regular

potentials, the sum of of a negative regular potential and a negative δ -measure supported by a curve etc. Of interest is the question if corollary 4 still holds for the case when the support of m has zero Lebesgue measure, as the above arguments do not work in that case.

In [7] we have found an interesting condition permitting to handle a class of singular perturbations without assumptions on the Lebesgue measure of the support.

Corollary 5. *Let $\nu \leq 0$ and the intersection of $\text{supp } \nu$ with a certain circle be an infinite set. Then H has at least $\#S$ eigenvalues below κ . In particular, this holds if m is spherically symmetric and $h < 0$.*

Proof. In [7] it is shown that under above assumptions the matrix $(\hat{\nu}(p_j - p_k))$ is negative definite for any choice of pairwise different p_j . \square

4 Perturbations supported by curves

The above results allows for analysis of perturbations supported by curves only in rather particular symmetric cases. We consider in greater detail perturbations supported by curves in this section.

Lemma 6. *Assume that*

- (a) *S contains a C^1 arc which is not an interval;*
- (b) *m is compactly supported;*
- (c) *$h(x) < 0$ for a.e. x with respect to the measure m ;*
- (d) *The Fourier transform $\hat{\nu} \equiv \widehat{hm}$ vanishes at infinity at least along a certain straight line, i.e. $\hat{\nu}(r \cos \alpha, r \sin \alpha) \rightarrow 0$ for $r \rightarrow \pm\infty$ and some $\alpha \in [0, 2\pi)$.*

Then the assumptions of Theorem 3 are satisfied for any N .

We prove first a simple lemma.

Lemma 7. *Let C be a C^1 arc in the plane \mathbb{R}^2 different from an interval. Then the set $X = \{p - q : p, q \in C\} \subset \mathbb{R}^2$ has an interior point.*

Proof. Let $t \mapsto (x(t), y(t))$, $t \in [0, 1]$, is a regular parametrization of C . Consider the mapping $f : (s, t) \mapsto (x(t) - x(s), y(t) - y(s))$, $s, t \in [0, 1]$. Clear, $X = f([0, 1] \times [0, 1])$. Since C is not an interval, the Jacobian

$$\det \begin{pmatrix} -x'(s) & -y'(s) \\ x'(t) & y'(t) \end{pmatrix}$$

does not vanish at a certain point $(s_0, t_0) \in (0, 1) \times (0, 1)$. By the implicit function theorem, f is invertible near (s_0, t_0) , therefore the image of some neighborhood of (s_0, t_0) is an open set. \square

Proof of Lemma 6. The assumption (b) implies the analyticity of $\hat{\nu}$. We show that for any N there are points $p_j \in S$, $j = 1, \dots, N$, such that the matrix $\mathcal{V}(p_1, \dots, p_n)$ with entries $\mathcal{V}_{jk} = \hat{\nu}(p_j - p_k)$ is negative definite. By the Bochner

theorem, \mathcal{V} is negative semi-definite for any choice of p_j , and it remains to show that $\det \mathcal{V}(p_1, \dots, p_N) \neq 0$ for a choice of points p_j from S .

Now we proceed by induction on N . For $N = 1$ one has, by (c), $\det \mathcal{V} = \hat{\nu}(0) < 0$.

Let now $N \geq 2$. Assume that there exist $p'_j \in S$, $j = 1, \dots, N-1$ such that $\det \mathcal{V}(p'_1, \dots, p'_{N-1}) \neq 0$.

Note that $\mathcal{V}(p_1, \dots, p_N)$ is a function of $N-1$ variables q_j , $q_j := p_j - p_N$, $j = 1, \dots, N-1$, $q_j \in X$, $X = \{p - q : p, q \in S\}$, $\mathcal{V}(p_1, \dots, p_N) = \tilde{\mathcal{V}}(q_1, \dots, q_{N-1})$, and is analytic in q_j . Hence, if $\det \mathcal{V}(p_1, \dots, p_N) = 0$ for any choice of $p_j \in S$, then, by Lemma 7, $\det \tilde{\mathcal{V}}(q_1, \dots, q_{N-1})$ vanishes on an open set and, due to the analyticity, $\det \mathcal{V}(p_1, \dots, p_N) = 0$ for any N -tuple (p_1, \dots, p_N) in $(\mathbb{R}^2)^N$.

On the other hand, by (c) we have $\mathcal{V}_{jj} = \hat{\nu}(0) < 0$ independently on (p_1, \dots, p_N) for every j . Using the determinant expansion and the condition (d) we obtain $\det \mathcal{V}(p'_1, \dots, p'_{N-1}, p_N) \rightarrow \hat{\nu}(0) \det \mathcal{V}(p'_1, \dots, p'_{N-1}) \neq 0$ as $p_N = r(\cos \alpha, \sin \alpha)$ and $r \rightarrow \pm\infty$. This contradiction concludes the proof. \square

The item (d) in Lemma 6 is, of course, the finest one among the assumptions, and there is an extended literature discussing the decay rates of the Fourier transforms of measures, see e. g. [1, 9, 10]. For our purposes it is sufficient to consider the following two cases.

Lemma 8. *Let Γ be a compact C^2 curve with non-vanishing curvature, and $\nu = \delta_\Gamma$, then the assumption (d) in Lemma 6 is satisfied.*

Proof. Let $[0, l] \ni t \mapsto (x(t), y(t))$ be the natural parametrization of Γ . As the curvature does not vanish, one has

$$|x'(t)y''(t) - x''(t)y'(t)| \geq c > 0 \quad \text{for all } t. \quad (15)$$

Consider the expression

$$\hat{\nu}(p) = \frac{1}{2\pi} \int_0^l \exp \left[-ir(\cos \alpha x(t) + \sin \alpha y(t)) \right] dt, \quad p = r(\cos \alpha, \sin \alpha).$$

By (15), there exists $\delta > 0$ for which one can divide Γ into several smooth pieces such that one of the following conditions is satisfied on each of them:

- $|\cos \alpha x'(t) + \cos \alpha y'(t)| \geq \delta$,
- $|\cos \alpha x''(t) + \cos \alpha y''(t)| \geq \delta$.

Hence, by the van der Corput lemma on oscillatory integrals, [15, Section VIII.1, Proposition 2], one has $\hat{\nu}(p) \rightarrow 0$ for $r \rightarrow \infty$. \square

Lemma 9. *Let Γ be a line segment, and $\nu = \delta_\Gamma$, then the assumption (d) in Lemma 6 is satisfied.*

Proof. Let $[0, 1] \ni t \mapsto (x_0 + at, y_0 + bt)$ be a parametrization of Γ . Writing

$$\begin{aligned} \hat{\nu}(p) &= \frac{\sqrt{a^2 + b^2}}{2\pi} \int_0^1 \exp \left[-ir(\cos \alpha(x_0 + at) + \sin \alpha(y_0 + bt)) \right] dt \\ &= \frac{\sqrt{a^2 + b^2} \exp(-ir(x_0 \cos \alpha + y_0 \sin \alpha))}{2\pi} \int_0^1 \exp \left[-irt(a \cos \alpha + b \sin \alpha) \right] dt, \\ &\quad p = r(\cos \alpha, \sin \alpha) \end{aligned}$$

we immediately see that $\hat{\nu}(r \cos \alpha, r \sin \alpha) \rightarrow 0$ as $r \rightarrow \pm\infty$ for any α with $a \cos \alpha + b \sin \alpha \neq 0$ by the Riemann-Lebesgue theorem. \square

Now we can use Lemmas 6, 8, and 9 to prove a general result on singular interactions supported by curves.

Theorem 10. *Assume that S contains a C^1 arc which is not a line segment. Let Γ be a C^2 curve, $m = \delta_\Gamma$, and the restriction of h to Γ be a negative continuous function, then the operator H has infinitely many eigenvalues below the essential spectrum.*

Proof. There exists a part of Γ , Γ' , with the following properties:

- Γ' is a compact C^2 curve,
- Γ' either has non-vanishing curvature or is a line segment,
- $h|_{\Gamma'} \leq -c$, $c > 0$.

Represent $\nu = \nu_1 + \nu_2$, $\nu_1 = -c\delta_{\Gamma'}$, $\nu_2 := \nu - \nu_1$. By construction one has $\nu_2 \leq 0$. By Lemma 6, for any N there exist $p_1, \dots, p_N \in S$ such that the matrix $(\hat{\nu}_1(p_j - p_k))$ is negative definite. As ν_2 is non-positive, the matrix $(\hat{\nu}_2(p_j - p_k))$ is at least negative semi-definite by the Bochner theorem, and $(\hat{\nu}(p_j - p_k))$ is hence negative definite. Thus, H has infinitely many eigenvalues below the essential spectrum by Theorem 3. \square

The assumption (a) of Lemma 6 obviously holds for the Rashba and Dresselhaus Hamiltonians. Hence, Theorem 10 guarantees the existence of infinitely many eigenvalues below the continuous spectrum under negative perturbations supported by smooth curves.

If the set S for the unperturbed Hamiltonian H_0 is very “bad” and does not contain any smooth arc, then an analogue of Lemma 7 becomes a difficult problem from the general topology. Nevertheless, the assumptions of Lemma 7 are naturally satisfied in reasonable examples including the above Rashba and Dresselhaus Hamiltonians.

We note in conclusion that the method proposed for estimating the number of eigenvalues is quite universal but very rough; it does not take into account e.g. the Kramers degeneracy (all eigenvalues of perturbed Rashba Hamiltonians must be at least twice degenerate).

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